

Question 10

Let

$$R = \{(x, y) : x \geq 0, y \geq 0, 1 \leq x + y \leq 3\}.$$

Use

$$u = x + y, \quad v = \frac{x}{x + y}$$

to evaluate $\iint_R (x + y)^{-1} dA$.

Tasks

1. Derive the inverse map and transformed rectangle.
2. Compute the Jacobian and integral.
3. Explain geometrically what the coordinate v measures.

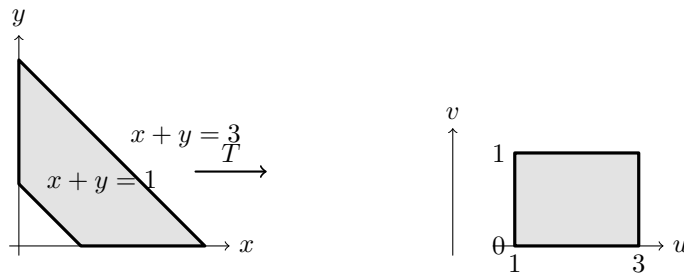
Question 10 – Solution

Strategy. Use total distance along the coordinate-sum direction and the fraction allocated to x .

Step 1: Inverse and geometry Since $u = x + y > 0$,

$$x = uv, \quad y = u(1 - v).$$

Nonnegativity gives $0 \leq v \leq 1$, while the two diagonal boundaries give $1 \leq u \leq 3$.



Step 2: Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} v & u \\ 1 - v & -u \end{vmatrix} = -u, \quad dA = u \, du \, dv.$$

Hence the integrand $1/(x + y) = 1/u$ cancels the area scale:

$$I = \int_1^3 \int_0^1 \frac{1}{u}(u) \, dv \, du = \boxed{2}.$$

Step 3: Interpretation and check On a segment $x + y = u$, the quantity $v = x/u$ is the fraction of the total coordinate sum assigned to x ; it runs once from the y -axis to the x -axis. Thus the inverse is one-to-one in the interior, and $u > 0$ confirms the absolute Jacobian used above.