

Question 8

The unit cube $0 \leq u, v, w \leq 1$ is mapped into xyz -space by

$$x = u + v, \quad y = v + w, \quad z = w + u.$$

Evaluate $\iiint_E (x + y + z) dV$, where E is the image parallelepiped.

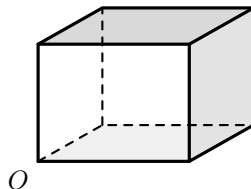
Tasks

1. Compute the three-dimensional Jacobian and establish invertibility.
2. Evaluate the transformed integral.
3. Verify using the image volume and centroid.

Question 8 – Solution

Strategy. A nonsingular linear map sends the unit cube to a parallelepiped with constant volume scale.

Step 1: Jacobian and geometry



$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2.$$

Because $J \neq 0$, the linear map is invertible and covers its image once.

Step 2: Transform the integrand We have $x + y + z = 2(u + v + w)$ and $dV = 2 du dv dw$. Therefore

$$\begin{aligned} I &= \int_0^1 \int_0^1 \int_0^1 4(u + v + w) dw dv du \\ &= 4 \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) = \boxed{6}. \end{aligned}$$

Verification The image volume is $|J|$ times the cube volume, hence 2. The cube centroid $(1/2, 1/2, 1/2)$ maps to $(1, 1, 1)$, where $x + y + z = 3$. A linear function averages to its centroid value, so $I = (2)(3) = 6$.