

Question 6

Consider the map

$$T(u, v) = (u^2, v), \quad -1 \leq u \leq 1, \quad 0 \leq v \leq 1.$$

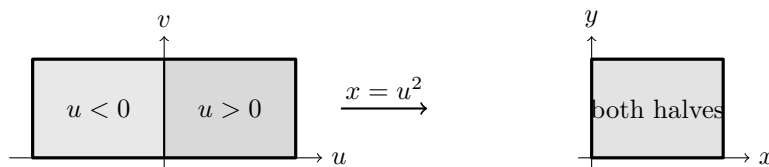
Tasks

1. Describe the image and determine whether T is one-to-one.
2. Explain why integrating $|J_T|$ over the full source gives the wrong image area.
3. Repair the change of variables calculation.

Question 6 – Solution

Strategy. Track multiplicity: the two halves of the source fold onto the same target rectangle.

Step 1: Image and folding



The image is $0 \leq x \leq 1$, $0 \leq y \leq 1$. Except along $u = 0$, the points (u, v) and $(-u, v)$ have the same image, so T is two-to-one.

Step 2: Diagnose the overcount Since

$$J_T = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2u & 0 \\ 0 & 1 \end{vmatrix} = 2u,$$

integrating its absolute value over the full source gives

$$\int_{-1}^1 \int_0^1 |2u| \, dv \, du = 2,$$

twice the actual image area 1. The absolute value corrects orientation, not multiplicity.

Step 3: Repair Restrict to the one-to-one half $0 \leq u \leq 1$:

$$\text{area}(T(S)) = \int_0^1 \int_0^1 2u \, dv \, du = \boxed{1}.$$

Using $-1 \leq u \leq 0$ alone gives the same result after taking $|J_T|$.