

Question 10

For $a > 0$, a uniform spherical shell occupies

$$a \leq \rho \leq 2a.$$

Its average distance from the origin is $45/7$.

Tasks

1. Derive the shell volume as a function of a .
2. Derive the average value of ρ .
3. Recover a , the outer radius, and the shell volume.

Question 10 – Solution

Strategy. Divide the first radial moment by volume; the common angular factors cancel, leaving a linear function of a .

Step 1: Volume

$$\begin{aligned} V(a) &= \int_0^{2\pi} \int_0^\pi \int_a^{2a} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= 4\pi \left[\frac{\rho^3}{3} \right]_a^{2a} = \frac{28\pi a^3}{3}. \end{aligned}$$

Step 2: Average radius The first radial moment is

$$M_\rho(a) = 4\pi \int_a^{2a} \rho^3 \, d\rho = 15\pi a^4.$$

Hence

$$\rho_{\text{avg}} = \frac{M_\rho(a)}{V(a)} = \frac{45a}{28}.$$

Step 3: Recover the shell Setting $45a/28 = 45/7$ gives $\boxed{a = 4}$. Therefore

$$\boxed{\text{outer radius} = 8}, \quad \boxed{V = \frac{1792\pi}{3}}.$$

Verification The average $45/7 \approx 6.43$ lies strictly between the recovered radii 4 and 8. Substituting $a = 4$ into $28\pi a^3/3$ gives the stated volume.