

Question 9

Explain why spherical coordinates use

$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta,$$

then evaluate $\iiint_E \rho^{-2} \, dV$ on the shell $1 \leq \rho \leq 3$.

Tasks

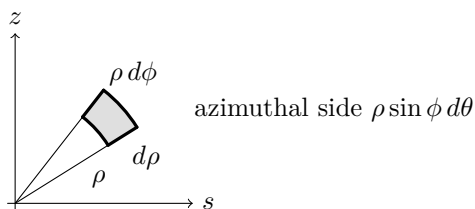
1. Derive the three local scale factors geometrically.
2. Evaluate the integral over the full shell.
3. Check the dimensional scaling and behavior near the origin.

Question 9 – Solution

Strategy. Approximate a small spherical cell by a box with one radial side and two angular arc-length sides.

Step 1: Volume element The local side lengths are

$$d\rho, \quad \rho d\phi, \quad \rho \sin \phi d\theta.$$



Their product is

$$(d\rho)(\rho d\phi)(\rho \sin \phi d\theta) = \rho^2 \sin \phi d\rho d\phi d\theta.$$

Step 2: Apply

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^\pi \int_1^3 \rho^{-2} \rho^2 \sin \phi d\rho d\phi d\theta \\ &= (2\pi)(2)(3-1) = \boxed{8\pi}. \end{aligned}$$

Verification The shell excludes $\rho = 0$, so the integrand is bounded. Its units are length^{-2} ; multiplying by volume produces length, matching the radial factor $3 - 1$ in the result. Even if the inner radius tends to 0, the radial integral remains finite: $4\pi \int_0^3 d\rho = 12\pi$. Thus the singularity is integrable.