

Question 8

A full spherical shell $1 \leq \rho \leq 2$ has density $\delta = \rho$. Find its moment of inertia about the z -axis.

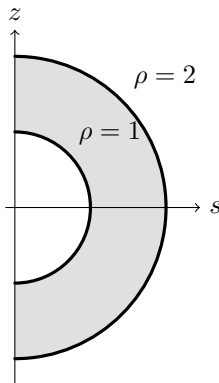
Tasks

1. Express the squared distance to the z -axis in spherical coordinates.
2. Compute the mass and I_z .
3. Find and check the radius of gyration $k_z = \sqrt{I_z/M}$.

Question 8 – Solution

Strategy. The distance to the z -axis is $\rho \sin \phi$; combine its square with the radial density and spherical Jacobian.

Step 1: Region and mass



$$M = \int_0^{2\pi} \int_0^\pi \int_1^2 \rho \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = (2\pi)(2) \frac{15}{4} = \boxed{15\pi}.$$

Step 2: Moment of inertia

$$\begin{aligned} I_z &= \iiint_E (\rho \sin \phi)^2 \delta \, dV \\ &= \int_0^{2\pi} \int_0^\pi \int_1^2 \rho^5 \sin^3 \phi \, d\rho \, d\phi \, d\theta \\ &= (2\pi) \left(\frac{4}{3} \right) \left(\frac{2^6 - 1}{6} \right) = \boxed{28\pi}. \end{aligned}$$

Step 3: Radius of gyration

$$\boxed{k_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{28}{15}}}.$$

It lies between the minimum and maximum distances 0 and 2 from the axis, as required.