

Question 7

Let

$$E = \{(x, y, z) : 2 \leq \sqrt{x^2 + y^2 + z^2} \leq 3, x \geq 0, y \geq x, z \leq 0\}.$$

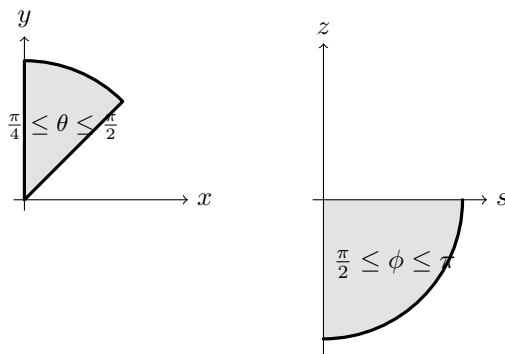
Tasks

1. Determine exact spherical bounds for E .
2. Find its volume.
3. Verify the angular fractions geometrically.

Question 7 – Solution

Strategy. Resolve the horizontal inequalities into a θ -wedge and the vertical inequality into a ϕ -hemisphere.

Step 1: Angular bounds The conditions $x \geq 0$ and $y \geq x$ select $\pi/4 \leq \theta \leq \pi/2$. The condition $z \leq 0$ selects $\pi/2 \leq \phi \leq \pi$, while the shell gives $2 \leq \rho \leq 3$.



Step 2: Volume

$$\begin{aligned} V &= \int_{\pi/4}^{\pi/2} \int_{\pi/2}^{\pi} \int_2^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \left(\frac{\pi}{4}\right) (1) \left(\frac{27-8}{3}\right) = \boxed{\frac{19\pi}{12}}. \end{aligned}$$

Verification The azimuthal wedge is $(\pi/4)/(2\pi) = 1/8$ of a full turn, and $z \leq 0$ supplies half the polar solid angle. Thus E occupies $1/16$ of the full shell; indeed $\frac{1}{16} \frac{4\pi}{3} (27-8) = 19\pi/12$.