

Question 6

Use spherical coordinates to find the volume inside

$$x^2 + y^2 + z^2 \leq 2z.$$

Tasks

1. Convert the boundary to a radial spherical bound.
2. Evaluate the volume.
3. Identify the Cartesian center and radius as a check.

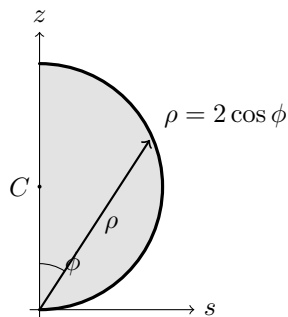
Question 6 – Solution

Strategy. The shifted sphere becomes $\rho \leq 2 \cos \phi$, so its nonconstant radial bound encodes the displacement from the origin.

Step 1: Bounds Substitution gives

$$\rho^2 \leq 2\rho \cos \phi.$$

For $\rho \geq 0$, the nontrivial radial interval is $0 \leq \rho \leq 2 \cos \phi$, which requires $0 \leq \phi \leq \pi/2$. Also $0 \leq \theta \leq 2\pi$.



Step 2: Evaluate

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2 \cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \frac{16\pi}{3} \int_0^{\pi/2} \cos^3 \phi \sin \phi \, d\phi = \boxed{\frac{4\pi}{3}}. \end{aligned}$$

Verification Completing the square gives $x^2 + y^2 + (z - 1)^2 \leq 1$, a sphere of radius 1 centered at $(0, 0, 1)$. Its ordinary volume is $4\pi/3$, confirming the integral.