

Question 5

Inside a ball of radius R , consider the cone sector

$$0 \leq \rho \leq R, \quad 0 \leq \phi \leq \alpha, \quad 0 \leq \theta \leq 2\pi,$$

where $0 < \alpha < \pi/2$. Its volume is one-third of the upper half-ball.

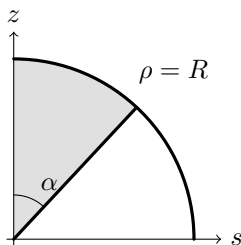
Tasks

1. Determine α .
2. Write the boundary cone in Cartesian coordinates.
3. Verify the stated volume ratio.

Question 5 – Solution

Strategy. The radial and azimuthal factors cancel from the ratio, leaving only the polar-angle factor $1 - \cos \alpha$.

Step 1: Sector volume



$$V_{\text{sector}} = \int_0^{2\pi} \int_0^\alpha \int_0^R \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{2\pi R^3}{3} (1 - \cos \alpha).$$

The upper half-ball has volume $2\pi R^3/3$. Hence

$$1 - \cos \alpha = \frac{1}{3}, \quad \boxed{\alpha = \arccos\left(\frac{2}{3}\right)}.$$

Step 2: Cartesian cone On $\phi = \alpha$, $z = s \cot \alpha$, where $s = \sqrt{x^2 + y^2}$. Since $\sin \alpha = \sqrt{5}/3$, we have $\cot \alpha = 2/\sqrt{5}$. Thus

$$\boxed{z = \frac{2}{\sqrt{5}} \sqrt{x^2 + y^2}}.$$

Verification Substituting $\cos \alpha = 2/3$ makes the angular factor $1 - \cos \alpha = 1/3$, so the sector volume is exactly one-third of the upper half-ball volume.