

Question 3

The upper half of the spherical shell $1 \leq \rho \leq 2$ has density $\delta = \rho$.

Tasks

1. Find its mass in spherical coordinates.
2. Find its average density.
3. Verify the average against the density range.

Question 3 – Solution

Strategy. The hemisphere contributes the full azimuthal interval but only $0 \leq \phi \leq \pi/2$; the radial density adds one power of ρ .

Step 1: Mass

$$\begin{aligned} M &= \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 \rho \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= (2\pi)(1) \left[\frac{\rho^4}{4} \right]_1^2 = \boxed{\frac{15\pi}{2}}. \end{aligned}$$

Step 2: Average density The region volume is half the difference of two ball volumes:

$$V = \frac{1}{2} \frac{4\pi}{3} (2^3 - 1^3) = \frac{14\pi}{3}.$$

Therefore

$$\boxed{\delta_{\text{avg}} = \frac{M}{V} = \frac{45}{28}}.$$

Verification Since $1 \leq \delta = \rho \leq 2$ throughout the shell, the value $45/28 \approx 1.61$ is admissible. The larger outer spherical layers correctly pull it above the midpoint $3/2$.