

Question 2

Find the volume inside the sphere $x^2 + y^2 + z^2 \leq 16$ and above the cone

$$z = \sqrt{3}\sqrt{x^2 + y^2}.$$

Tasks

1. Convert the cone to a constant ϕ -boundary.
2. Set up and evaluate the spherical integral.
3. Compare with the volume of the containing ball.

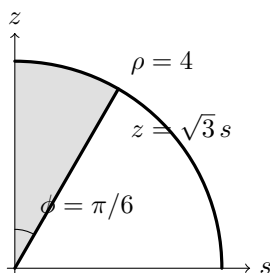
Question 2 – Solution

Strategy. In a meridian plane, the cone fixes the polar angle while the sphere fixes the radial endpoint.

Step 1: Angular boundary Since $z = \rho \cos \phi$ and $\sqrt{x^2 + y^2} = \rho \sin \phi$,

$$\rho \cos \phi = \sqrt{3} \rho \sin \phi \implies \tan \phi = \frac{1}{\sqrt{3}}.$$

The region above the cone therefore has $0 \leq \phi \leq \pi/6$.



Step 2: Evaluate

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\pi/6} \int_0^4 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= 2\pi \left(1 - \frac{\sqrt{3}}{2} \right) \frac{64}{3} = \boxed{\frac{64\pi}{3} (2 - \sqrt{3})}. \end{aligned}$$

Verification The angular factor $1 - \sqrt{3}/2$ is between 0 and 1, so this positive sector is smaller than the upper half-ball and therefore smaller than the entire ball of volume $256\pi/3$.