

Question 1

Let E be the part of the spherical shell

$$1 \leq \sqrt{x^2 + y^2 + z^2} \leq 3$$

in the first octant. Evaluate

$$\iiint_E \frac{1}{\sqrt{x^2 + y^2 + z^2}} dV$$

using spherical coordinates.

Tasks

1. Convert the region, integrand, and volume element.
2. Evaluate the integral.
3. Check the result using the integrand range and region volume.

Question 1 – Solution

Strategy. A first-octant spherical shell has constant bounds in all three spherical variables, and the radial integrand cancels one power of the Jacobian.

Step 1: Convert Using

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi,$$

the bounds are

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{2}, \quad 1 \leq \rho \leq 3,$$

and $dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$.

Step 2: Evaluate

$$\begin{aligned} I &= \int_0^{\pi/2} \int_0^{\pi/2} \int_1^3 \frac{1}{\rho} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \left(\frac{\pi}{2}\right) (1) \left[\frac{\rho^2}{2}\right]_1^3 = \boxed{2\pi}. \end{aligned}$$

Verification The shell-octant volume is $\frac{1}{8} \frac{4\pi}{3} (3^3 - 1^3) = 13\pi/3$. Hence the average integrand is $I/V = 6/13$, which lies between its endpoint values $1/3$ and 1 .