

Question 10

For $a > 0$, let E_a be the solid

$$0 \leq z \leq a^2, \quad z \geq x^2 + y^2.$$

Its average z -coordinate is 6.

Tasks

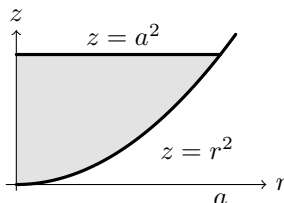
1. Express E_a in cylindrical coordinates.
2. Derive its volume and average z -coordinate as functions of a .
3. Recover a and the corresponding volume.

Question 10 – Solution

Strategy. The horizontal plane meets the paraboloid at $r = a$; compute both volume and first z -moment before imposing the average.

Step 1: Bounds The solid lies above $z = r^2$ and below $z = a^2$:

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq a, \quad r^2 \leq z \leq a^2.$$



Step 2: Volume and moment

$$V(a) = 2\pi \int_0^a (a^2 - r^2)r \, dr = \frac{\pi a^4}{2},$$
$$M_z(a) = 2\pi \int_0^a \int_{r^2}^{a^2} zr \, dz \, dr = \frac{\pi a^6}{3}.$$

Therefore

$$z_{\text{avg}} = \frac{M_z(a)}{V(a)} = \frac{2a^2}{3}.$$

Step 3: Recover the parameter The condition $2a^2/3 = 6$ gives $a^2 = 9$. Since $a > 0$,

$$\boxed{a = 3}, \quad \boxed{V(3) = \frac{81\pi}{2}}.$$

Verification For $a = 3$, the solid has $0 \leq z \leq 9$, and the average 6 lies inside that range. Substitution returns $2(3^2)/3 = 6$ exactly.