

Question 9

Explain why cylindrical coordinates use

$$dV = r \, dz \, dr \, d\theta,$$

then apply the formula to evaluate $\iiint_E (x^2 + y^2) \, dV$, where

$$1 \leq r \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{3}, \quad 0 \leq z \leq 4.$$

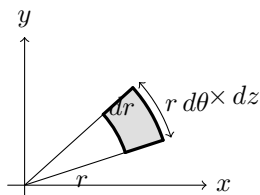
Tasks

1. Derive the scale factor r geometrically.
2. Evaluate the integral.
3. Verify the dimensional scaling.

Question 9 – Solution

Strategy. Approximate a small cylindrical cell by a rectangular box whose angular side is an arc of length $r d\theta$.

Step 1: Jacobian factor A small cell has radial length dr , vertical length dz , and circumferential length approximately $r d\theta$.



Consequently,

$$dV \approx (dr)(r d\theta)(dz) = r dz dr d\theta,$$

and the approximation becomes exact in the integral limit.

Step 2: Apply Since $x^2 + y^2 = r^2$,

$$\begin{aligned} I &= \int_0^{\pi/3} \int_1^2 \int_0^4 r^2 (r dz dr d\theta) \\ &= \left(\frac{\pi}{3}\right) (4) \left[\frac{r^4}{4}\right]_1^2 = \boxed{5\pi}. \end{aligned}$$

Verification The factors scale as angular width times height times radius to the fourth power. That produces length^5 , the correct units for integrating a squared distance over volume.