

Question 7

Let E be the ball $x^2 + y^2 + z^2 \leq 4$. Without using spherical coordinates, evaluate

$$\iiint_E (x^2 + y^2) \, dV.$$

Tasks

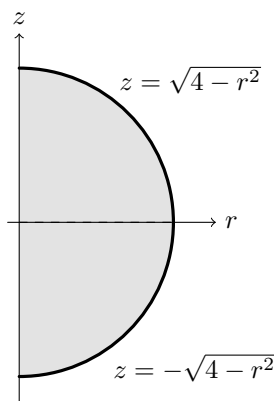
1. Express the ball with cylindrical bounds.
2. Evaluate the integral exactly.
3. Check positivity and magnitude.

Question 7 – Solution

Strategy. At fixed cylindrical radius r , the sphere supplies symmetric upper and lower z -bounds.

Step 1: Bounds Since $r^2 + z^2 \leq 4$,

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 2, \quad -\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2}.$$



Step 2: Evaluate

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^2 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^2 (r \, dz \, dr \, d\theta) \\ &= 4\pi \int_0^2 r^3 \sqrt{4-r^2} \, dr. \end{aligned}$$

With $u = 4 - r^2$, $r^3 dr = -(4 - u)du/2$, so

$$\int_0^2 r^3 \sqrt{4-r^2} \, dr = \frac{1}{2} \int_0^4 (4-u)u^{1/2} du = \frac{64}{15}.$$

Thus $\boxed{I = 256\pi/15}$.

Verification The integrand is nonnegative and at most 4. Since the ball volume is $32\pi/3$, the required bound $0 < I < 128\pi/3$ holds.