

Question 10

Let D be the upper half-disk $x^2 + y^2 \leq 9$, $y \geq 0$. Evaluate

$$\iint_D (x^2 + y^2 + xy) dA.$$

Tasks

1. Convert the region and integrand to polar coordinates.
2. Use angular symmetry to eliminate the mixed term.
3. Evaluate the remaining integral and verify its sign.

Question 10 – Solution

Strategy. In polar form the radial term is r^2 , while $xy = r^2 \cos \theta \sin \theta$ cancels over $0 \leq \theta \leq \pi$.

Step 1: Set up

$$0 \leq \theta \leq \pi, \quad 0 \leq r \leq 3,$$

so

$$I = \int_0^\pi \int_0^3 r^3 (1 + \cos \theta \sin \theta) \, dr \, d\theta.$$

Step 2: Symmetry Since

$$\int_0^\pi \cos \theta \sin \theta \, d\theta = 0,$$

the xy contribution vanishes, matching reflection symmetry across the y -axis.

Step 3: Evaluate

$$I = \int_0^\pi \left[\frac{r^4}{4} \right]_0^3 d\theta = \int_0^\pi \frac{81}{4} d\theta = \boxed{\frac{81\pi}{4}}.$$

The surviving integrand $x^2 + y^2$ is nonnegative, so the positive result is necessary. Its scale is radius squared times half-disk area, as expected.