

Question 8

For $k > 0$, let D_k be the region enclosed by $y = x^2$ and $y = kx$.

Tasks

1. Find the intersections and write bounds for D_k .
2. Derive a formula for the area as a function of k .
3. Determine the unique $k > 0$ for which the area is $9/2$.

Question 8 – Solution

Strategy. Keep the line slope symbolic; the nonzero intersection becomes the outer endpoint of a vertical-slice area integral.

Step 1: Intersections and bounds From $x^2 = kx$,

$$x(x - k) = 0,$$

so the intersections occur at $x = 0, k$. Since $k > 0$, $x^2 \leq kx$ on $[0, k]$, and

$$D_k = \{(x, y) : 0 \leq x \leq k, x^2 \leq y \leq kx\}.$$

Step 2: Area formula

$$\begin{aligned} A(k) &= \int_0^k \int_{x^2}^{kx} 1 \, dy \, dx = \int_0^k (kx - x^2) \, dx \\ &= \left[\frac{kx^2}{2} - \frac{x^3}{3} \right]_0^k = \boxed{\frac{k^3}{6}}. \end{aligned}$$

Step 3: Inverse condition Set $k^3/6 = 9/2$. Then $k^3 = 27$, and the positive restriction gives

$$\boxed{k = 3}.$$

Substitution yields $A(3) = 27/6 = 9/2$. Since $A(k) = k^3/6$ is strictly increasing for $k > 0$, this positive solution is unique.