

Question 7

A solid lies above the triangle

$$D = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}$$

and below the plane $z = 1 - x - y$.

Tasks

1. Write Cartesian bounds for D .
2. Evaluate the volume using a double integral.
3. Verify the answer using the geometry of a tetrahedron.

Question 7 – Solution

Strategy. Use vertical slices of the triangular base, then compare the integral with the intercept-form tetrahedron.

Step 1: Bounds For $0 \leq x \leq 1$, the variable y runs from 0 to $1 - x$. Therefore

$$V = \int_0^1 \int_0^{1-x} (1 - x - y) dy dx.$$

Step 2: Evaluate

$$\begin{aligned} V &= \int_0^1 \left[(1-x)y - \frac{y^2}{2} \right]_0^{1-x} dx \\ &= \frac{1}{2} \int_0^1 (1-x)^2 dx = \frac{1}{2} \left[x - x^2 + \frac{x^3}{3} \right]_0^1 \\ &= \boxed{\frac{1}{6}}. \end{aligned}$$

Step 3: Geometric check The coordinate planes and $x + y + z = 1$ enclose a tetrahedron with mutually perpendicular intercept lengths 1, 1, 1. Its volume is

$$\frac{1}{3}(\text{base area})(\text{height}) = \frac{1}{3} \left(\frac{1}{2} \right) (1) = \frac{1}{6}.$$

This matches the integral and verifies the top surface remains nonnegative over D .