

Question 10

Over the square $R = [-1, 1] \times [-1, 1]$, a solid lies between

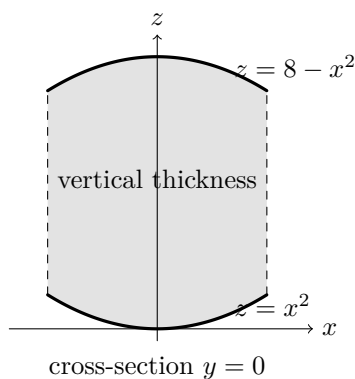
$$z_{\text{top}} = 8 - x^2 - y^2 \quad \text{and} \quad z_{\text{bottom}} = x^2 + y^2.$$

Tasks

1. Verify that the named top surface stays above the bottom surface on all of R .
2. Set up and evaluate an iterated integral for the volume.
3. Check the answer using symmetry and separate one-variable integrals.

Question 10 – Solution

Strategy. Integrate vertical thickness over the rectangular base, after confirming that the thickness never changes sign.



Step 1: Thickness The vertical thickness is

$$h(x, y) = 8 - 2x^2 - 2y^2.$$

On R , $x^2 + y^2 \leq 2$, so $h \geq 8 - 4 = 4 > 0$. Thus no top-bottom reversal or absolute value is needed.

Step 2: Iterated integral

$$\begin{aligned} V &= \int_{-1}^1 \int_{-1}^1 (8 - 2x^2 - 2y^2) dy dx \\ &= \int_{-1}^1 \left(16 - 4x^2 - \frac{4}{3} \right) dx \\ &= \left[\frac{44}{3}x - \frac{4}{3}x^3 \right]_{-1}^1 = \boxed{\frac{80}{3}}. \end{aligned}$$

Step 3: Independent check The constant term contributes $8 \text{ area}(R) = 32$. Also,

$$\iint_R x^2 dA = \left(\int_{-1}^1 x^2 dx \right) (2) = \frac{4}{3},$$

and identically $\iint_R y^2 dA = 4/3$. Hence

$$V = 32 - 2 \left(\frac{4}{3} \right) - 2 \left(\frac{4}{3} \right) = 32 - \frac{16}{3} = \frac{80}{3},$$

confirming the iterated calculation.