

Question 9

Evaluate

$$\iint_R (x^3 \cos y + x^2 \sin y + x^2) dA, \quad R = [-2, 2] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Tasks

1. Integrate with respect to y first and identify the term eliminated by y -symmetry.
2. Use x -symmetry in the outer integral.
3. Evaluate the surviving contribution and verify its sign.

Question 9 – Solution

Strategy. Use parity at each stage: the symmetric inner interval cancels the sine term, and the symmetric outer interval cancels the remaining odd power of x .

Step 1: Inner y -integration

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} (x^3 \cos y + x^2 \sin y + x^2) dy &= x^3 [\sin y]_{-\pi/2}^{\pi/2} - x^2 [\cos y]_{-\pi/2}^{\pi/2} + \pi x^2 \\ &= 2x^3 + \pi x^2.\end{aligned}$$

The $x^2 \sin y$ term vanishes because sine is odd in y .

Step 2: Outer x -integration Now $2x^3$ is odd on $[-2, 2]$, so it contributes zero. Therefore

$$\begin{aligned}\iint_R f dA &= \pi \int_{-2}^2 x^2 dx = \pi \left[\frac{x^3}{3} \right]_{-2}^2 \\ &= \boxed{\frac{16\pi}{3}}.\end{aligned}$$

Step 3: Verification The only surviving term is $x^2 \geq 0$, integrated over a y -interval of length π . Thus a positive result is necessary. Its one-variable integral is $16/3$, so multiplication by π confirms the scale and constant.