

Question 8

Let p be continuous on $[a, b]$ and q continuous on $[c, d]$.

Tasks

1. Prove using iterated integrals that

$$\iint_{[a,b] \times [c,d]} p(x)q(y) \, dA = \left(\int_a^b p(x) \, dx \right) \left(\int_c^d q(y) \, dy \right).$$

2. Explain why the same proof works in either order.
3. Apply the result to $(1 + x^2) \cos y$ on $[-1, 1] \times [-\pi/2, \pi/2]$.

Question 8 – Solution

Strategy. In the inner integral, the factor depending only on the outer variable is constant and can be pulled outside.

Step 1: Factorization proof Integrating in y first gives

$$\begin{aligned}\int_a^b \int_c^d p(x)q(y) dy dx &= \int_a^b p(x) \left(\int_c^d q(y) dy \right) dx \\ &= \left(\int_a^b p(x) dx \right) \left(\int_c^d q(y) dy \right).\end{aligned}$$

The bracketed q -integral is a number, so it is constant in the outer x -integration.

Step 2: Other order Reversing the order first produces the constant $\int_a^b p(x) dx$ and then the q -integral. Continuity on the closed rectangle guarantees Fubini's theorem applies, so both forms represent the same double integral.

Step 3: Application Here

$$\int_{-1}^1 (1+x^2) dx = \left[x + \frac{x^3}{3} \right]_{-1}^1 = \frac{8}{3}$$

and

$$\int_{-\pi/2}^{\pi/2} \cos y dy = [\sin y]_{-\pi/2}^{\pi/2} = 2.$$

Therefore

$$\boxed{\iint_R (1+x^2) \cos y dA = \frac{16}{3}}.$$

Both factors are nonnegative on their intervals, verifying the sign.