

Question 6

For a real parameter a , define

$$I(a) = \int_0^2 \int_{-1}^1 (axy^2 + x^2 - 2y) \, dy \, dx.$$

Tasks

1. Evaluate $I(a)$ exactly.
2. Find the unique value of a for which the integrand's average value on the rectangle is 1.
3. Verify the resulting integral directly from the inner accumulation function.

Question 6 – Solution

Strategy. Use parity in the symmetric y -interval, retain the parameter symbolically, and convert the prescribed average into a total integral.

Step 1: Inner accumulation Since $-2y$ is odd,

$$\begin{aligned}\int_{-1}^1 (axy^2 + x^2 - 2y) dy &= ax \left[\frac{y^3}{3} \right]_{-1}^1 + x^2[y]_{-1}^1 \\ &= \frac{2ax}{3} + 2x^2.\end{aligned}$$

Thus

$$I(a) = \int_0^2 \left(\frac{2ax}{3} + 2x^2 \right) dx = \frac{4a}{3} + \frac{16}{3}.$$

Step 2: Prescribed average The rectangle's area is $2 \cdot 2 = 4$. Average value 1 therefore requires $I(a) = 4$:

$$\frac{4a + 16}{3} = 4 \implies \boxed{a = -1}.$$

Step 3: Direct verification With $a = -1$, the inner accumulation is

$$-\frac{2x}{3} + 2x^2.$$

Consequently,

$$\int_0^2 \left(-\frac{2x}{3} + 2x^2 \right) dx = \left[-\frac{x^2}{3} + \frac{2x^3}{3} \right]_0^2 = -\frac{4}{3} + \frac{16}{3} = 4.$$

Dividing by area 4 returns the required average 1.