

**Question 9**

Find the volume of the ellipsoidal shell

$$1 \leq \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} \leq 4.$$

Use the change of variables

$$x = 2u, \quad y = 3v, \quad z = 4w.$$

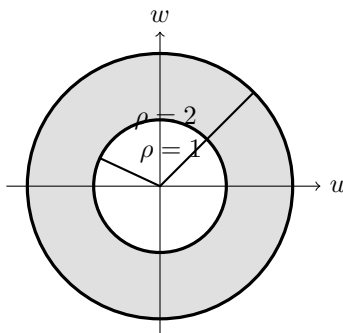
**Tasks**

1. Describe the transformed shell.
2. Compute the three-dimensional Jacobian and the volume.
3. Check the scaling factor geometrically.

### Question 9 – Solution

**Strategy.** The coordinate scaling turns the ellipsoids into concentric spheres and multiplies every volume by a constant Jacobian.

**Step 1: Transform the shell**



scaled spherical shell

The inequality becomes

$$1 \leq u^2 + v^2 + w^2 \leq 4,$$

so the transformed region is the spherical shell  $1 \leq \rho \leq 2$ .

**Step 2: Jacobian and volume** The derivative matrix is diagonal, so

$$\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = |(2)(3)(4)| = 24.$$

Consequently,

$$V = 24 \left[ \frac{4\pi}{3} (2^3 - 1^3) \right] = 24 \left( \frac{28\pi}{3} \right) = \boxed{224\pi}.$$

**Verification** The map stretches lengths by factors 2, 3, and 4 along the coordinate axes, so it must stretch volume by their product 24, exactly the Jacobian used above.