

Question 7

Consider the solids

$$S_1 = \{(x, y, z) : 0 \leq z \leq 1, x^2 + y^2 \leq z^2\}$$

and

$$S_2 = \{(x, y, z) : 0 \leq z \leq 1, 0 \leq x \leq z^2, 0 \leq y \leq \pi\}.$$

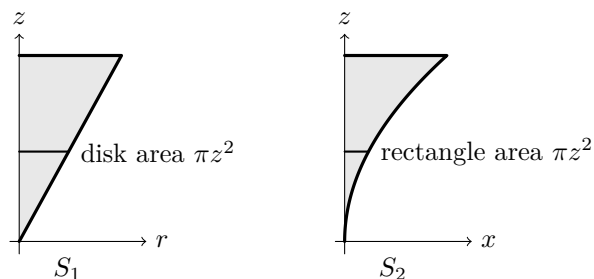
Tasks

1. Find the area of each horizontal cross-section.
2. Use Cavalieri's principle to compare the volumes.
3. Compute their common volume.

Question 7 – Solution

Strategy. Although the slices have different shapes, compare their areas at the same height.

Step 1: Slice areas



At height z , the slice of S_1 is a disk of radius z , so

$$A_1(z) = \pi z^2.$$

The slice of S_2 is a rectangle of side lengths z^2 and π , so

$$A_2(z) = z^2\pi = \pi z^2.$$

Step 2: Compare and integrate Both solids occupy $0 \leq z \leq 1$ and have equal cross-sectional area at every height. Cavalieri's principle therefore gives equal volumes, and

$$V_1 = V_2 = \int_0^1 \pi z^2 dz = \pi \left[\frac{z^3}{3} \right]_0^1 = \boxed{\frac{\pi}{3}}.$$

Verification The conclusion depends on slice areas, not slice shapes: circular slices in S_1 and rectangular slices in S_2 accumulate the same volume height by height.