

Question 6

Find the volume of the solid torus

$$\left(\sqrt{x^2 + y^2} - 3\right)^2 + z^2 \leq 1$$

using cylindrical coordinates.

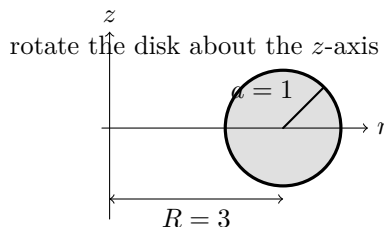
Tasks

1. Describe the bounds in the meridian (r, z) -plane.
2. Evaluate the triple integral, using symmetry where helpful.
3. Give a geometric check of the answer.

Question 6 – Solution

Strategy. In cylindrical coordinates the torus is obtained by rotating a radius-1 disk centered at $(r, z) = (3, 0)$.

Step 1: Bounds



The radial variable ranges from 2 to 4, and

$$-\sqrt{1 - (r - 3)^2} \leq z \leq \sqrt{1 - (r - 3)^2}, \quad 0 \leq \theta \leq 2\pi.$$

Step 2: Evaluate With $u = r - 3$,

$$\begin{aligned} V &= 2\pi \int_2^4 2r \sqrt{1 - (r - 3)^2} dr \\ &= 4\pi \int_{-1}^1 (u + 3) \sqrt{1 - u^2} du. \end{aligned}$$

The u -term is odd, while $\int_{-1}^1 \sqrt{1 - u^2} du = \pi/2$. Therefore

$$V = 4\pi(3) \frac{\pi}{2} = \boxed{6\pi^2}.$$

Verification The generating disk has area π , and its center travels a circle of circumference $2\pi(3) = 6\pi$. Their product is $6\pi^2$, agreeing with the integral.