

Question 5

A cylindrical hole of radius a is drilled through the center of a sphere of radius R , where $0 < a < R$. The remaining solid has height

$$h = 2\sqrt{R^2 - a^2}.$$

Find its volume and express the result using only h .

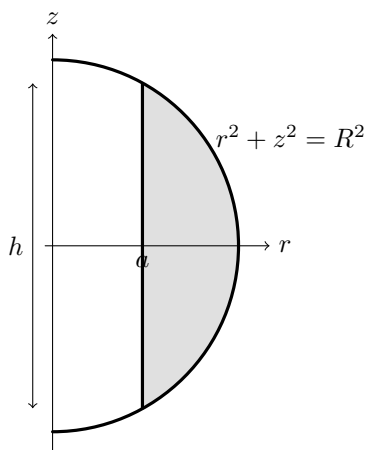
Tasks

1. Set up the volume in cylindrical coordinates.
2. Evaluate it and eliminate R and a in favor of h .
3. Explain the resulting “napkin-ring” invariance.

Question 5 – Solution

Strategy. Use cylindrical shells with $a \leq r \leq R$; the sphere supplies the upper and lower z -bounds.

Step 1: Bounds



The sphere is $r^2 + z^2 \leq R^2$. Hence

$$0 \leq \theta \leq 2\pi, \quad a \leq r \leq R, \quad -\sqrt{R^2 - r^2} \leq z \leq \sqrt{R^2 - r^2}.$$

Step 2: Evaluate

$$\begin{aligned} V &= \int_0^{2\pi} \int_a^R 2\sqrt{R^2 - r^2} r \, dr \, d\theta \\ &= 4\pi \left[-\frac{(R^2 - r^2)^{3/2}}{3} \right]_a^R = \frac{4\pi}{3} (R^2 - a^2)^{3/2}. \end{aligned}$$

Since $R^2 - a^2 = h^2/4$,

$$\boxed{V = \frac{\pi h^3}{6}}.$$

Verification The final formula contains only the remaining height. Thus every drilled sphere with the same height h leaves the same volume, regardless of its original R and hole radius a .