

Question 4

Let $a > 0$. Two perpendicular circular cylinders form the solid

$$x^2 + z^2 \leq a^2, \quad y^2 + z^2 \leq a^2.$$

Find its volume.

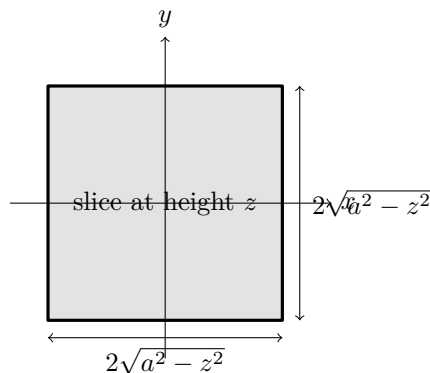
Tasks

1. Describe a horizontal cross-section at a fixed height z .
2. Integrate the cross-sectional area.
3. Check symmetry and dimensional scaling.

Question 4 – Solution

Strategy. At each fixed z , the two cylinder inequalities give independent bounds on x and y , producing a square.

Step 1: Cross-section



For $-a \leq z \leq a$,

$$|x| \leq \sqrt{a^2 - z^2}, \quad |y| \leq \sqrt{a^2 - z^2}.$$

Thus the slice is a square with area

$$A(z) = \left(2\sqrt{a^2 - z^2}\right)^2 = 4(a^2 - z^2).$$

Step 2: Integrate

$$\begin{aligned} V &= \int_{-a}^a 4(a^2 - z^2) dz = 8 \int_0^a (a^2 - z^2) dz \\ &= 8 \left[a^2 z - \frac{z^3}{3} \right]_0^a = \boxed{\frac{16a^3}{3}}. \end{aligned}$$

Verification The cross-sectional area is even in z , justifying the factor 2. The answer is proportional to a^3 , the correct scaling for a volume.