

Question 1

Let $a > 0$. Find the area enclosed by the astroid

$$x^{2/3} + y^{2/3} = a^{2/3}$$

using the change of variables $x = u^3$, $y = v^3$.

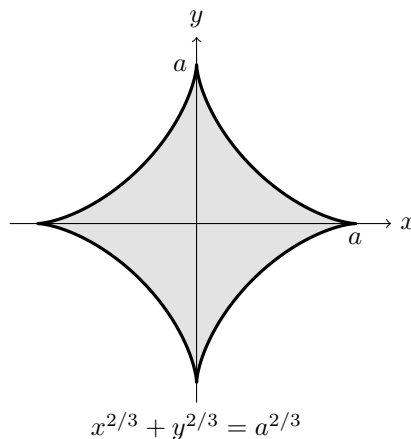
Tasks

1. Describe the transformed region and justify the substitution.
2. Compute the Jacobian and evaluate the area.
3. Check the scaling of the answer with respect to a .

Question 1 – Solution

Strategy. The cubic substitution converts the astroid into a disk, after which polar coordinates handle the transformed integral.

Step 1: Transform the region



Because the real cube function is one-to-one,

$$x^{2/3} = u^2, \quad y^{2/3} = v^2.$$

Thus the interior maps one-to-one to $u^2 + v^2 \leq a^{2/3}$, a disk of radius $a^{1/3}$.

Step 2: Jacobian and integral The absolute Jacobian is

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = |(3u^2)(3v^2)| = 9u^2v^2.$$

Writing $u = r \cos \theta$, $v = r \sin \theta$ gives

$$\begin{aligned} A &= 9 \int_0^{2\pi} \int_0^{a^{1/3}} r^5 \cos^2 \theta \sin^2 \theta \, dr \, d\theta \\ &= 9 \left(\frac{a^2}{6} \right) \left(\frac{\pi}{4} \right) = \boxed{\frac{3\pi a^2}{8}}. \end{aligned}$$

Verification Replacing a by ka enlarges every length by k , so area must scale by k^2 . The result is proportional to a^2 , as required.