

Question 10

A function f on the unit square satisfies the Lipschitz estimate

$$|f(P) - f(Q)| \leq 2\|P - Q\|$$

for all points P, Q . Let S_n be the midpoint Riemann sum on an $n \times n$ uniform grid.

Tasks

1. Prove the error bound $|\iint_R f \, dA - S_n| \leq \sqrt{2}/n$.
2. Find the least n certified by this bound to give error less than 0.01.
3. If $S_{142} = 2.731$, give a guaranteed interval containing the true integral.

Question 10 – Solution

Strategy. Control the variation within each cell by its maximum distance from the midpoint, then add the cellwise error bounds.

Step 1: One-cell error The Lipschitz estimate makes f continuous, hence integrable on the closed square. A grid square has side $1/n$, area $1/n^2$, and maximum distance from its midpoint to a corner

$$d_n = \frac{\sqrt{2}}{2n}.$$

If M_{ij} is its midpoint, the Lipschitz condition gives

$$|f(P) - f(M_{ij})| \leq 2d_n = \frac{\sqrt{2}}{n}$$

throughout that cell. Its integral differs from midpoint height times area by at most $\sqrt{2}/n^3$.

Step 2: Global error There are n^2 cells, so the triangle inequality yields

$$\left| \iint_R f \, dA - S_n \right| \leq n^2 \frac{\sqrt{2}}{n^3} = \frac{\sqrt{2}}{n}.$$

To guarantee strict error below 0.01, require $n > 100\sqrt{2} \approx 141.421$. The least integer is therefore n=142.

Step 3: Numerical enclosure For $n = 142$,

$$\frac{\sqrt{2}}{142} \approx 0.009959.$$

Thus

$$2.731 - \frac{\sqrt{2}}{142} \leq \iint_R f \, dA \leq 2.731 + \frac{\sqrt{2}}{142},$$

or approximately

$$2.72104 \leq \iint_R f \, dA \leq 2.74096.$$

The half-diagonal, rather than the full cell diagonal, is essential to the stated constant.