

Question 9

Define the bounded function on the unit square by

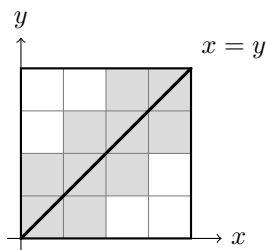
$$f(x, y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases}$$

Tasks

1. Explain why every lower sum is zero.
2. Use a uniform $n \times n$ grid to bound an upper sum by $3/n$.
3. Prove that f is Riemann integrable and find its double integral, despite its discontinuities.

Question 9 – Solution

Strategy. The diagonal has no area: trap every possible nonzero upper-sum contribution in a shrinking collection of grid squares.



Step 1: Lower sums Every subrectangle with positive area contains a point off the diagonal, where $f = 0$. Its infimum is therefore 0, so every lower sum equals 0.

Step 2: Upper-sum cover In an $n \times n$ grid, each square has area $1/n^2$. The n diagonal squares meet $x = y$. At each of the $n - 1$ interior diagonal grid vertices, two additional off-diagonal squares have closures meeting the diagonal. Thus at most

$$n + 2(n - 1) = 3n - 2$$

closed grid squares have supremum 1; all others have supremum 0. Consequently,

$$0 \leq U_n \leq \frac{3n - 2}{n^2} < \frac{3}{n}.$$

Step 3: Integrability Given $\varepsilon > 0$, choose $n > 3/\varepsilon$. Then

$$0 \leq U_n - L_n < \frac{3}{n} < \varepsilon.$$

The Darboux criterion proves integrability, and the integral is squeezed between zero lower sums and upper sums tending to zero:

$$\boxed{\iint_{[0,1]^2} f \, dA = 0}.$$

The function is discontinuous at every point of the diagonal, but that one-dimensional set contributes no area.