

Question 8

An affine function $f(x, y) = a + bx + cy$ is defined on the unit square. Its average value is 4 on the whole square, 5 on the right half $[1/2, 1] \times [0, 1]$, and 3 on the top half $[0, 1] \times [1/2, 1]$.

Tasks

1. Use center values of affine functions to obtain three equations for a, b, c .
2. Determine f uniquely.
3. Predict the integral of f over the lower-left quarter $[0, 1/2] \times [0, 1/2]$ and verify consistency.

Question 8 – Solution

Strategy. The average of an affine function over a rectangle equals its value at that rectangle's center; use the three given centers to reconstruct the coefficients.

Step 1: Average equations The relevant centers are $(1/2, 1/2)$, $(3/4, 1/2)$, and $(1/2, 3/4)$. Hence

$$\begin{aligned}a + \frac{b}{2} + \frac{c}{2} &= 4, \\a + \frac{3b}{4} + \frac{c}{2} &= 5, \\a + \frac{b}{2} + \frac{3c}{4} &= 3.\end{aligned}$$

The center-value rule follows from the same reflection pairing used for any affine graph over a rectangle.

Step 2: Reconstruct Subtracting the first equation from the second gives $b/4 = 1$, so $b = 4$. Subtracting the first from the third gives $c/4 = -1$, so $c = -4$. The first equation then gives $a = 4$. Therefore

$$\boxed{f(x, y) = 4 + 4x - 4y}.$$

Step 3: New prediction The lower-left quarter has center $(1/4, 1/4)$ and area $1/4$. Its average is

$$f\left(\frac{1}{4}, \frac{1}{4}\right) = 4 + 1 - 1 = 4,$$

so

$$\boxed{\iint_{[0, 1/2]^2} f \, dA = 4 \left(\frac{1}{4}\right) = 1}.$$

The equal x - and y -center coordinates cancel the opposite slopes, which checks the predicted average.