

Question 6

On the centrally symmetric rectangle

$$R = [-2, 2] \times [-1, 1],$$

consider $f(x, y) = 5 + x^3 + xy^2 + 2y$.

Tasks

1. Show that the nonconstant part cancels under central reflection.
2. Evaluate $\iint_R f \, dA$ and find the average value of f .
3. Explain precisely which symmetry of the region makes the argument valid.

Question 6 – Solution

Strategy. Pair each point (x, y) with its central reflection $(-x, -y)$ and average the two function values.

Step 1: Pairing Let $h(x, y) = x^3 + xy^2 + 2y$. Then

$$h(-x, -y) = -x^3 - xy^2 - 2y = -h(x, y).$$

Thus the contributions of h at every reflected pair cancel. Equivalently,

$$f(x, y) + f(-x, -y) = 10.$$

Step 2: Integral and average Central reflection maps R onto itself without changing area. Therefore the paired average of f is 5 everywhere, and

$$\iint_R f \, dA = 5 \operatorname{area}(R) = 5(4)(2) = \boxed{40}.$$

Dividing by the area 8 gives

$$\boxed{f_{\text{avg}} = 5}.$$

Step 3: Symmetry check The required property is

$$(x, y) \in R \iff (-x, -y) \in R.$$

Symmetry in only one coordinate would not automatically cancel the entire h : xy^2 is odd in x , while $2y$ is odd in y . Central symmetry reverses both coordinates at once and cancels all three terms.