

Question 3

Let $f(x, y) = xy$ on the unit square $R = [0, 1] \times [0, 1]$. Partition R into an $n \times n$ uniform grid.

Tasks

1. Compute the lower sum L_n and upper sum U_n .
2. Use them to determine $\iint_R xy \, dA$ without iterated integration.
3. Find the least integer n for which the gap $U_n - L_n$ is less than 0.01.

Question 3 – Solution

Strategy. Because xy increases in each coordinate on the unit square, lower-left and upper-right corners produce the lower and upper sums.

Step 1: Lower sum Every cell has area $1/n^2$. Using indices $i, j = 0, \dots, n-1$ at lower-left corners,

$$\begin{aligned} L_n &= \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \frac{i}{n} \frac{j}{n} = \frac{1}{n^4} \left(\sum_{i=0}^{n-1} i \right)^2 \\ &= \frac{1}{n^4} \left(\frac{n(n-1)}{2} \right)^2 = \frac{(n-1)^2}{4n^2}. \end{aligned}$$

Step 2: Upper sum and squeeze Upper-right corners give

$$U_n = \frac{1}{n^4} \left(\sum_{i=1}^n i \right)^2 = \frac{(n+1)^2}{4n^2}.$$

Both sums approach $1/4$. Since $L_n \leq \iint_R xy \, dA \leq U_n$,

$$\boxed{\iint_R xy \, dA = \frac{1}{4}}.$$

Step 3: Resolution The uncertainty is

$$U_n - L_n = \frac{(n+1)^2 - (n-1)^2}{4n^2} = \frac{1}{n}.$$

Thus $1/n < 0.01$ requires $n > 100$, and the least integer is n=101. The strict inequality rules out $n = 100$.