

Question 2

On $R = [0, 1] \times [0, 2]$, let $f(x, y) = x + 2y$. Divide the x -interval into m equal parts and the y -interval into n equal parts. Use the upper-right corner of each subrectangle as its sample point.

Tasks

1. Write the resulting double Riemann sum $S_{m,n}$.
2. Simplify the finite sum exactly using summation formulas.
3. Take the limit as $m, n \rightarrow \infty$ to evaluate the double integral from its definition.

Question 2 – Solution

Strategy. Express every sample point and area in terms of its indices, then separate the finite sums algebraically before taking the two-parameter limit.

Step 1: Grid and samples We have

$$\Delta x = \frac{1}{m}, \quad \Delta y = \frac{2}{n}, \quad \Delta A = \frac{2}{mn},$$

and the upper-right sample in cell (i, j) is $(i/m, 2j/n)$. Thus

$$S_{m,n} = \sum_{i=1}^m \sum_{j=1}^n \left(\frac{i}{m} + \frac{4j}{n} \right) \frac{2}{mn}.$$

Step 2: Exact finite sum Using $\sum_{k=1}^N k = N(N+1)/2$,

$$\begin{aligned} S_{m,n} &= \frac{2}{m^2 n} \sum_{i=1}^m \sum_{j=1}^n i + \frac{8}{mn^2} \sum_{i=1}^m \sum_{j=1}^n j \\ &= \frac{2}{m^2 n} \left(n \frac{m(m+1)}{2} \right) + \frac{8}{mn^2} \left(m \frac{n(n+1)}{2} \right) \\ &= \frac{m+1}{m} + 4 \frac{n+1}{n}. \end{aligned}$$

Step 3: Definition limit Both correction terms vanish independently, so

$$\boxed{\iint_R (x + 2y) dA = \lim_{m,n \rightarrow \infty} S_{m,n} = 1 + 4 = 5}.$$

The answer lies between $0 \cdot 2$ and $5 \cdot 2 = 10$, the bounds obtained from the extreme values of f on the area-2 rectangle.