

Question 10

Find the absolute extrema of

$$f(x, y, z) = xyz$$

on the ellipsoid

$$x^2 + 2y^2 + 3z^2 = 6.$$

Tasks

1. Solve the Lagrange equations for candidates with nonzero product.
2. Treat zero-coordinate candidates and compare all possible values.
3. List every maximizing and minimizing point and verify the sign pattern.

Question 10 – Solution

Strategy. Multiplying each component equation by its corresponding coordinate reveals fixed ratios among x^2, y^2, z^2 .

Step 1: Nonzero-product candidates The multiplier system is

$$yz = 2\lambda x, \quad xz = 4\lambda y, \quad xy = 6\lambda z.$$

When $xyz \neq 0$, multiply these equations by x, y, z , respectively:

$$xyz = 2\lambda x^2 = 4\lambda y^2 = 6\lambda z^2.$$

Thus

$$2x^2 = 4y^2 = 6z^2 = K.$$

Substitution into the constraint gives

$$\frac{K}{2} + 2\frac{K}{4} + 3\frac{K}{6} = \frac{3K}{2} = 6,$$

so $K = 4$. Therefore

$$x^2 = 2, \quad y^2 = 1, \quad z^2 = \frac{2}{3}.$$

Step 2: Values and zero cases At these eight sign choices,

$$|xyz| = \sqrt{2} \cdot 1 \cdot \sqrt{\frac{2}{3}} = \frac{2}{\sqrt{3}}.$$

Any feasible candidate with a zero coordinate has $f = 0$, which lies strictly between the positive and negative nonzero values.

Step 3: All extrema The ellipsoid is compact, so comparison is decisive:

$$f_{\max} = \frac{2}{\sqrt{3}}$$

at the four points $(\pm\sqrt{2}, \pm 1, \pm\sqrt{2/3})$ having an even number of minus signs, and

$$f_{\min} = -\frac{2}{\sqrt{3}}$$

at the four such points having an odd number of minus signs. The parity rule follows because the product is positive for an even number of negative factors and negative for an odd number.