

Question 8

For the family

$$f_a(x, y) = x^2 + ay^2,$$

the point $P = (1, 2)$ is a constrained extremum on the line $x + y = 3$.

Tasks

1. Determine the parameter a using the Lagrange condition.
2. Classify the constrained extremum.
3. Verify the classification by substituting the constraint into the resulting function.

Question 8 – Solution

Strategy. Parallel gradients at the specified regular point determine the parameter; a one-variable restriction then establishes the classification.

Step 1: Recover a For $g = x + y$,

$$\nabla f_a(1, 2) = \langle 2, 4a \rangle, \quad \nabla g = \langle 1, 1 \rangle.$$

The equation $\langle 2, 4a \rangle = \lambda \langle 1, 1 \rangle$ gives $\lambda = 2$ and $4a = 2$. Hence

$$\boxed{a = \frac{1}{2}}.$$

Step 2: Restrict to the line Set $y = 3 - x$. Then

$$f_{1/2}(x, 3 - x) = x^2 + \frac{1}{2}(3 - x)^2.$$

Completing the square,

$$x^2 + \frac{1}{2}(3 - x)^2 = \frac{3}{2}(x - 1)^2 + 3.$$

Therefore the constrained value is minimized uniquely at $x = 1$, $y = 2$.

Step 3: Result and verification

$$\boxed{P = (1, 2) \text{ is a strict constrained minimum}}, \quad \boxed{f_{1/2}(1, 2) = 3}.$$

The restricted quadratic tends to infinity as $|x| \rightarrow \infty$, so the minimum is also absolute along the entire constraint line; no constrained maximum exists.