

Question 7

Find the constrained extrema of

$$f(x, y) = x^2 + y^2$$

subject to

$$x^2 + xy + y^2 = 3.$$

Tasks

1. Solve the Lagrange equations without assuming x or y is nonzero.
2. Determine all maximizing and minimizing points and values.
3. Verify the result by rewriting the constraint in rotated directions $y = x$ and $y = -x$.

Question 7 – Solution

Strategy. Eliminate the multiplier by combining the two component equations; the resulting factorization identifies the principal directions.

Step 1: Multiplier equations

$$\langle 2x, 2y \rangle = \lambda \langle 2x + y, x + 2y \rangle, \quad x^2 + xy + y^2 = 3.$$

Multiply the first component equation by y , the second by x , and subtract. The left sides cancel, leaving

$$\lambda(y^2 - x^2) = 0.$$

The constraint rules out $(0, 0)$, and $\lambda = 0$ would force $x = y = 0$. Hence

$$y = \pm x.$$

Step 2: Candidate values If $y = x$, the constraint is $3x^2 = 3$, so $(x, y) = (1, 1)$ or $(-1, -1)$ and $f = 2$.

If $y = -x$, the constraint is $x^2 = 3$, giving $(\sqrt{3}, -\sqrt{3})$ and $(-\sqrt{3}, \sqrt{3})$ with $f = 6$. Therefore

$$f_{\min} = 2 \text{ at } (1, 1), (-1, -1),$$

$$f_{\max} = 6 \text{ at } (\sqrt{3}, -\sqrt{3}), (-\sqrt{3}, \sqrt{3}).$$

Step 3: Verification The constraint ellipse is compact. Along its $y = x$ principal direction its radius squared is 2, while along $y = -x$ its radius squared is 6. These are exactly the values of $x^2 + y^2$, confirming the classifications and completeness.