

Question 5

The sphere

$$x^2 + y^2 + z^2 = 9$$

meets the plane

$$x + y + z = 3.$$

Find the highest and lowest points of their intersection curve.

Tasks

1. Use two Lagrange multipliers to extremize the height $f(x, y, z) = z$.
2. Solve the complete system and identify both points.
3. Verify the result geometrically or by direct substitution.

Question 5 – Solution

Strategy. At a constrained extreme with two regular constraints, ∇f lies in the span of both constraint gradients.

Step 1: Two-multiplier system Let

$$g = x^2 + y^2 + z^2, \quad h = x + y + z.$$

Then

$$\langle 0, 0, 1 \rangle = \lambda \langle 2x, 2y, 2z \rangle + \mu \langle 1, 1, 1 \rangle.$$

The first two component equations imply $2\lambda(x - y) = 0$. If $\lambda = 0$, then the first equation gives $\mu = 0$, contradicting the third. Hence $x = y = a$.

Step 2: Enforce both constraints The plane gives $z = 3 - 2a$. Substitution into the sphere gives

$$2a^2 + (3 - 2a)^2 = 9,$$

or

$$6a(a - 2) = 0.$$

Thus $a = 0$, $z = 3$, or $a = 2$, $z = -1$. Therefore

$$\boxed{\text{highest point } (0, 0, 3)}, \quad \boxed{\text{lowest point } (2, 2, -1)}.$$

Step 3: Verification Both points satisfy the sphere and plane equations. Their heights are 3 and -1 . The intersection is a closed circle, hence compact; the two candidates exhaust the multiplier equations, so these are the absolute height extrema.