

Question 4

A rectangular box has positive side lengths x, y, z and fixed sum of its three side lengths

$$x + y + z = 12.$$

Find the dimensions giving the largest volume $V = xyz$.

Tasks

1. Solve the three-variable Lagrange multiplier system.
2. Find the maximum volume.
3. Verify global maximality by checking the boundary of the closed simplex and using an inequality.

Question 4 – Solution

Strategy. The multiplier equations force equal positive side lengths; then compare with the zero-volume boundary and verify by AM–GM.

Step 1: Lagrange system For $g = x + y + z$,

$$\nabla V = \langle yz, xz, xy \rangle = \lambda \langle 1, 1, 1 \rangle.$$

Because $x, y, z > 0$, equality of the three components gives

$$yz = xz = xy,$$

and division by positive factors yields $x = y = z$. The constraint then gives

$$\boxed{x = y = z = 4}.$$

Step 2: Volume

$$\boxed{V_{\max} = 4 \cdot 4 \cdot 4 = 64}.$$

Step 3: Global verification Allowing $x, y, z \geq 0$ closes the feasible simplex. Its boundary has at least one zero side, so $V = 0$ there. In the positive interior, AM–GM gives

$$\frac{x + y + z}{3} \geq (xyz)^{1/3}.$$

Since the left side is 4,

$$xyz \leq 64,$$

with equality only when $x = y = z$. Thus the Lagrange candidate is the unique positive absolute maximizer.