

Question 3

Find the points on the parabola

$$y = x^2$$

closest to $A = (0, 2)$.

Tasks

1. Formulate the squared-distance objective and the constraint.
2. Solve the Lagrange system, including all branches.
3. Compare candidates and justify that the reported minimum is absolute despite the noncompact constraint.

Question 3 – Solution

Strategy. Minimize squared distance, which has the same minimizers as distance but simpler derivatives.

Step 1: Set up Let

$$f(x, y) = x^2 + (y - 2)^2, \quad g(x, y) = y - x^2.$$

The system $\nabla f = \lambda \nabla g$ is

$$2x = -2\lambda x, \quad 2(y - 2) = \lambda, \quad y = x^2.$$

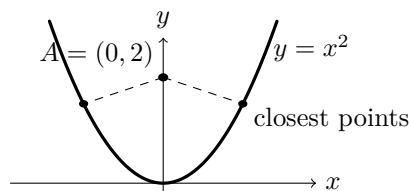
Step 2: Solve branches If $x = 0$, then $y = 0$, producing the candidate $(0, 0)$ with squared distance 4.

If $x \neq 0$, the first equation gives $\lambda = -1$. The second gives $2(y - 2) = -1$, so $y = 3/2$. Hence

$$x = \pm \sqrt{\frac{3}{2}}.$$

At either point the squared distance is

$$\frac{3}{2} + \left(-\frac{1}{2}\right)^2 = \boxed{\frac{7}{4}}.$$



Step 3: Conclusion Comparing candidates gives the two closest points

$$\boxed{\left(\pm \sqrt{\frac{3}{2}}, \frac{3}{2}\right)},$$

each at distance $\boxed{\sqrt{7}/2}$ from A . Along the parabola, $f(x, x^2) = x^2 + (x^2 - 2)^2 \rightarrow \infty$ as $|x| \rightarrow \infty$. Therefore no minimizing sequence can escape to infinity, and the smaller candidate value is the absolute minimum.