

Question 2

Find the maximum and minimum of

$$f(x, y) = xy$$

subject to the ellipse

$$x^2 + 4y^2 = 4.$$

Tasks

1. Solve the multiplier equations, treating possible zero coordinates carefully.
2. List all constrained extrema and values.
3. Verify the result with a substitution that converts the constraint to a circle.

Question 2 – Solution

Strategy. Solve the component equations without dividing until zero-coordinate cases have been checked.

Step 1: Multiplier equations With $g = x^2 + 4y^2$,

$$\langle y, x \rangle = \lambda \langle 2x, 8y \rangle, \quad x^2 + 4y^2 = 4.$$

If $x = 0$, the first multiplier equation forces $y = 0$, violating the constraint; similarly $y = 0$ is impossible. Thus $xy \neq 0$, and

$$y = 2\lambda x, \quad x = 8\lambda y.$$

Consequently $16\lambda^2 = 1$ and $y = \pm x/2$. The constraint then gives $2x^2 = 4$, so $x = \pm\sqrt{2}$.

Step 2: Candidates and values Equal signs for x and y give $xy = 1$; opposite signs give $xy = -1$. Therefore

$$f_{\max} = 1 \text{ at } (\sqrt{2}, \sqrt{2}/2), (-\sqrt{2}, -\sqrt{2}/2),$$

$$f_{\min} = -1 \text{ at } (\sqrt{2}, -\sqrt{2}/2), (-\sqrt{2}, \sqrt{2}/2).$$

Step 3: Verification Let $u = x$ and $v = 2y$. Then $u^2 + v^2 = 4$ and $f = uv/2$. Since

$$|uv| \leq \frac{u^2 + v^2}{2} = 2,$$

we have $|f| \leq 1$, with equality precisely when $|u| = |v|$. This reproduces all four points.