

Question 8

For a real parameter a , define

$$f_a(x, y) = (x - a)^2 + y^2$$

on the square $S = [-1, 1] \times [-1, 1]$.

Tasks

1. Determine the absolute minimum value and all minimizing points for every a .
2. Determine the absolute maximum value and all maximizing points for every a .
3. Treat the tie case $a = 0$ explicitly and interpret the answer as squared distance.

Question 8 – Solution

Strategy. The function is squared distance from (x, y) to $(a, 0)$; minimize and maximize the two nonnegative coordinate contributions independently.

Step 1: Minimum The y^2 term is minimized uniquely at $y = 0$. The closest $x \in [-1, 1]$ to a is the clipped value

$$x_* = \begin{cases} -1, & a < -1, \\ a, & -1 \leq a \leq 1, \\ 1, & a > 1. \end{cases}$$

Hence the unique minimizing point is $(x_*, 0)$, and

$$f_{\min} = (\max\{|a| - 1, 0\})^2.$$

Step 2: Maximum The largest possible y^2 is 1, attained at $y = \pm 1$. The endpoint of $[-1, 1]$ farthest from a is at distance $|a| + 1$. Therefore

$$f_{\max} = (|a| + 1)^2 + 1.$$

If $a > 0$, the maximizers are $(-1, \pm 1)$; if $a < 0$, they are $(1, \pm 1)$.

Step 3: Tie case When $a = 0$, both x -endpoints are equally far away, so all four corners $(\pm 1, \pm 1)$ maximize f_0 with value 2. The point $(0, 0)$ uniquely minimizes it with value 0. This agrees exactly with the squared-distance interpretation.