

Question 7

On the unbounded domain $\mathbb{R}^2$, consider

$$f(x, y) = x^2 + y^2 + \frac{1}{1 + x^2 + y^2}.$$

Tasks

1. Determine all absolute minima and their value.
2. Decide whether an absolute maximum exists.
3. Explain why this example shows that compactness is sufficient, but not necessary, for an absolute minimum to exist.

Question 7 – Solution

Strategy. Reduce the function to the radial variable $s = x^2 + y^2 \geq 0$ and study a one-variable function on an unbounded interval.

Step 1: Radial reduction Let

$$g(s) = s + \frac{1}{1+s}, \quad s \geq 0.$$

Then

$$g'(s) = 1 - \frac{1}{(1+s)^2}.$$

At $s = 0$, $g'(0) = 0$, and for every $s > 0$, $(1+s)^2 > 1$, so $g'(s) > 0$. Therefore g increases strictly away from $s = 0$.

It follows that

$$\boxed{f_{\min} = 1 \text{ at } (0,0)}.$$

Step 2: No maximum As $x^2 + y^2 = s \rightarrow \infty$,

$$g(s) = s + \frac{1}{1+s} \rightarrow \infty.$$

Thus f is unbounded above and has $\boxed{\text{no absolute maximum}}$.

Step 3: Interpretation The domain $\mathbb{R}^2$ is closed but not bounded, so the Extreme Value Theorem does not apply. Nevertheless, direct analysis proves that an absolute minimum is attained. Compactness guarantees both extrema for every continuous function, but a particular continuous function may still attain one or both extrema on a noncompact domain.