

Question 6

Let

$$D = \{(x, y) : x^2 + y^2 < 1\}$$

be the open unit disk, and let $f(x, y) = x + y$.

Tasks

1. Determine whether f has an absolute maximum or minimum on D .
2. Find its supremum and infimum and give sequences approaching them.
3. Explain exactly which hypothesis of the Extreme Value Theorem is absent.

Question 6 – Solution

Strategy. Bound the linear function with Cauchy–Schwarz, then determine whether equality points belong to the open disk.

Step 1: Bounds For $(x, y) \in D$,

$$|x + y| \leq \sqrt{x^2 + y^2} \sqrt{2} < \sqrt{2}.$$

Therefore every function value lies strictly between $-\sqrt{2}$ and $\sqrt{2}$.

Step 2: Approaching the bounds For $0 < r < 1$, the point

$$\left(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}} \right)$$

lies in D and gives $f = r\sqrt{2} \rightarrow \sqrt{2}$ as $r \rightarrow 1^-$. Its negative gives values tending to $-\sqrt{2}$. Thus

$$\boxed{\sup_D f = \sqrt{2}}, \quad \boxed{\inf_D f = -\sqrt{2}}.$$

Neither value is attained, so f has no absolute maximum or minimum on D .

Step 3: The theorem The function is continuous and the disk is bounded, but D is not closed: its boundary circle is omitted. Hence the domain is not compact, and the Extreme Value Theorem does not guarantee attained extrema. The missing equality points are precisely

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \quad \text{and} \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).$$