

Question 4

Find the absolute extrema of

$$f(x, y) = x + 2y$$

on and inside the ellipse

$$x^2 + 4y^2 \leq 4.$$

Do not use Lagrange multipliers.

Tasks

1. Transform the ellipse to a disk with a change of variables.
2. Use an inequality to derive sharp upper and lower bounds.
3. Find all equality points and verify they lie on the ellipse.

Question 4 – Solution

Strategy. Let $u = x$ and $v = 2y$; then the region is a radius-2 disk and the objective is $u + v$.

Step 1: Transform Set

$$u = x, \quad v = 2y.$$

The constraint becomes $u^2 + v^2 \leq 4$, while

$$f = u + v = \langle u, v \rangle \cdot \langle 1, 1 \rangle.$$

Step 2: Sharp bounds By Cauchy–Schwarz,

$$|u + v| \leq \sqrt{u^2 + v^2} \sqrt{2} \leq 2\sqrt{2}.$$

Thus $-2\sqrt{2} \leq f \leq 2\sqrt{2}$.

Equality on the upper side requires $\langle u, v \rangle$ to point with $\langle 1, 1 \rangle$ and have length 2, giving

$$(u, v) = (\sqrt{2}, \sqrt{2}).$$

Lower equality gives its negative.

Step 3: Return to (x, y) Because $x = u$ and $y = v/2$,

$$f_{\max} = 2\sqrt{2} \text{ at } \left(\sqrt{2}, \frac{\sqrt{2}}{2} \right),$$

$$f_{\min} = -2\sqrt{2} \text{ at } \left(-\sqrt{2}, -\frac{\sqrt{2}}{2} \right).$$

For either point, $x^2 + 4y^2 = 2 + 2 = 4$, verifying boundary membership and sharpness.