

Question 10

A local profit model, in suitable units, is

$$P(x, y) = 80x + 60y - 2x^2 - xy - y^2.$$

Assume the critical point found below lies in the interior of the operational region.

Tasks

1. Find the critical production levels.
2. Classify the critical point using the Hessian.
3. Complete the square in translated coordinates to verify the local classification and compute the modeled profit there.

Question 10 – Solution

Strategy. Solve the marginal-profit equations, then verify the Hessian conclusion by expressing every nearby change as a negative quadratic form.

Step 1: Critical production levels

$$P_x = 80 - 4x - y, \quad P_y = 60 - x - 2y.$$

The equations $4x + y = 80$ and $x + 2y = 60$ yield

$$\boxed{x = \frac{100}{7}, \quad y = \frac{160}{7}}.$$

Step 2: Hessian test

$$P_{xx} = -4, \quad P_{yy} = -2, \quad P_{xy} = -1.$$

Thus

$$D = (-4)(-2) - (-1)^2 = 7 > 0.$$

Because $P_{xx} < 0$, the critical point is a strict relative maximum. Direct substitution gives

$$\boxed{P\left(\frac{100}{7}, \frac{160}{7}\right) = \frac{8800}{7}}.$$

Step 3: Translated verification Let $h = x - 100/7$ and $k = y - 160/7$. Since P is quadratic and its linear terms vanish at the critical point,

$$P(x, y) - \frac{8800}{7} = -2h^2 - hk - k^2.$$

Complete the square:

$$-2h^2 - hk - k^2 = -2\left(h + \frac{k}{4}\right)^2 - \frac{7}{8}k^2 < 0$$

for every $(h, k) \neq (0, 0)$. This independently verifies the strict relative maximum.