

Question 9

Consider

$$f(x, y) = \sqrt{x^2 + y^2}.$$

Tasks

1. Prove directly that the origin is a strict relative minimum.
2. Determine whether $f_x(0, 0)$ and $f_y(0, 0)$ exist.
3. Explain how this example refines the usual critical-point search procedure.

Question 9 – Solution

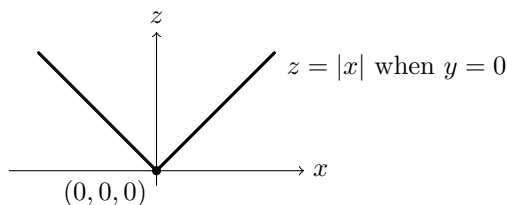
Strategy. Use the definition of a relative minimum first, then test each partial derivative with a signed two-sided limit.

Step 1: Direct extremum test At the origin, $f(0, 0) = 0$. For every $(x, y) \neq (0, 0)$,

$$f(x, y) = \sqrt{x^2 + y^2} > 0 = f(0, 0).$$

Therefore

$(0, 0)$ is a strict relative minimum.



Step 2: Partial derivatives Along the x -axis,

$$\frac{f(h, 0) - f(0, 0)}{h} = \frac{|h|}{h},$$

whose right-hand limit is 1 and left-hand limit is -1 . Thus $f_x(0, 0)$ does not exist. The identical calculation on the y -axis shows that $f_y(0, 0)$ does not exist.

Step 3: Methodological conclusion Fermat's theorem says that a *differentiable* interior extremum has zero gradient. It does not say every extremum is differentiable. Consequently, a complete candidate search must include interior points where first partial derivatives vanish *and* points where they fail to exist. This example supplies the latter case.