

Question 8

Let f be differentiable at an interior point $P = (a, b)$ of its domain.

Tasks

1. Prove that if f has a relative maximum or minimum at P , then $\nabla f(P) = \mathbf{0}$.
2. Explain why the converse is false.
3. Apply the result to show that $g(x, y) = x + e^y$ has no relative extrema on $\mathbb{R}^2$.

Question 8 – Solution

Strategy. Restrict the function to coordinate lines through the interior point and apply the one-variable necessary condition.

Step 1: Necessary condition Suppose f has a relative extremum at (a, b) . The one-variable restriction

$$\phi(x) = f(x, b)$$

has a relative extremum at $x = a$. Differentiability gives

$$f_x(a, b) = \phi'(a) = 0.$$

Likewise, $\psi(y) = f(a, y)$ has a relative extremum at b , so

$$f_y(a, b) = \psi'(b) = 0.$$

Therefore

$$\boxed{\nabla f(P) = \mathbf{0}}.$$

Step 2: The converse A zero gradient makes P a critical point, not necessarily an extremum. For instance, $f(x, y) = x^2 - y^2$ has gradient zero at the origin but takes both positive and negative values nearby, so the origin is a saddle.

Step 3: Application For $g(x, y) = x + e^y$,

$$g_x = 1, \quad g_y = e^y.$$

In particular, the gradient is never zero. Since every point of $\mathbb{R}^2$ is interior and g is differentiable everywhere, the necessary condition rules out a relative maximum or minimum at every point. Hence

$$\boxed{g \text{ has no relative extrema on } \mathbb{R}^2}.$$