

Question 7

A twice differentiable function has the local expansion near a critical point P :

$$f(P + (h, k)) = 5 + 3h^2 + 4hk - k^2 + o(h^2 + k^2).$$

Tasks

1. Reconstruct the Hessian matrix at P .
2. Apply the second derivative test.
3. Use the remainder term carefully to prove the classification along two coordinate directions.

Question 7 – Solution

Strategy. Match the quadratic terms with $\frac{1}{2}(h, k)H(h, k)^T$, then show that the leading signs dominate the smaller remainder.

Step 1: Hessian The second-order Taylor term is

$$\frac{1}{2} (f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2).$$

Matching coefficients gives

$$f_{xx} = 6, \quad f_{xy} = 4, \quad f_{yy} = -2,$$

so

$$H(P) = \begin{pmatrix} 6 & 4 \\ 4 & -2 \end{pmatrix}.$$

Step 2: Second derivative test

$$D = (6)(-2) - 4^2 = -28 < 0.$$

Therefore P is a saddle point.

Step 3: Remainder verification Along $k = 0$,

$$f(P + (h, 0)) - 5 = 3h^2 + o(h^2),$$

which is positive for all sufficiently small nonzero h . Along $h = 0$,

$$f(P + (0, k)) - 5 = -k^2 + o(k^2),$$

which is negative for all sufficiently small nonzero k . The little- o terms are eventually smaller in magnitude than, for example, half of the displayed quadratic terms. Thus nearby values occur on both sides of 5, rigorously confirming the saddle.