

Question 5

Let

$$f(x, y) = (x^2 + y^2)e^{-(x^2 + y^2)}.$$

Tasks

1. Find the complete critical set.
2. Classify the origin and every point on the critical circle as relative extrema.
3. Explain why the circular extrema are not strict and why the ordinary Hessian test degenerates there.

Question 5 – Solution

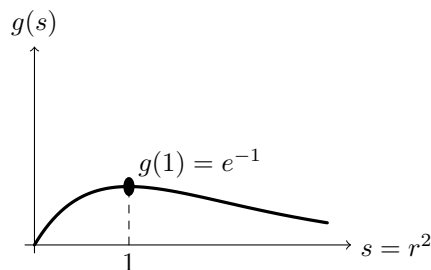
Strategy. Set $s = x^2 + y^2$ and analyze the one-variable radial profile $g(s) = se^{-s}$.

Step 1: Critical set Since

$$f_x = 2xe^{-s}(1-s), \quad f_y = 2ye^{-s}(1-s),$$

both derivatives vanish when $(x, y) = (0, 0)$ or when $s = 1$. Thus

$$\text{critical set} = \{(0, 0)\} \cup \{x^2 + y^2 = 1\}.$$



Step 2: Radial classification For $s \geq 0$,

$$g'(s) = e^{-s}(1-s).$$

It is positive for $0 \leq s < 1$ and negative for $s > 1$. Thus $g(0) = 0$ increases immediately away from the origin, so

$$(0, 0) \text{ is a strict relative minimum.}$$

The radial profile peaks at $s = 1$, so every point of the unit circle is a relative maximum with value

$$e^{-1}.$$

Step 3: Non-strictness and degeneracy Moving tangentially along the circle leaves s and f unchanged, so none of these maxima is strict. That zero-curvature tangential direction makes the Hessian determinant zero on the critical circle, which is why the standard second derivative test is inconclusive there.